EIGENMODES AND INTERFACE DESCRIPTION IN A HALL-HEROULT CELL

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ABSTRACT

In a series of papers, the authors have introduced a general method for calculating the stability of a Hall-Héroult cell with high accuracy. The stability of the fluid motions, and particularly the geometry of the interface between aluminum and bath, is modeled using a linearization of the magnetohydrodynamic equations around a steady-state solution of the full set of equations under the given operating conditions of the cell. Measurements were performed on an unstable cell, in which the currents in the 16 anode rods were recorded simultaneously. These measured currents are used as input disturbances to the stability model. The resulting slightly modified force field in turn excites the different modes of oscillation of the cell with amplitudes which are directly related to the amplitudes of the fluctuation of the current. The amplitudes of the different modes are thus determined, so that the time-dependent behavior of the different fields, in particular those of metal surface contour and of the velocity, can be described. A video-recording of the simulated metal surface will be presented

1. INTRODUCTION

The authors have been studying stability questions in a Hall-Héroult cell for many years. The modeling they are leaning on is obtained by a linearization of the MHD equations for incompressible fluids around a steady solution.

Each field is thus expressed as a sum of a steady part and a small time dependent one called the fluctuation. In this approach the study of stability is reduced to two different problems which consist in obtaining first, a steady state of the system and second, a linearized set of equations for the fluctuations.The time dependence of these fluctuations is assumed to be of the form $\exp(\lambda t)$ ($\lambda \in \mathbb{C}$ and $t \in \mathbb{R}$) so that one is led to solve a kind of generalized "eigenvalue" problem (with eigenvalue λ). In [5] it is shown that the solutions which are obtained by an analytic continuation, from the gravitational (with a non vanishing interface fluctuation) to the MHD modes, are eigensolutions of an unbounded operator in an appropriate function space. In other words to each eigenvalue of this operator corresponds eigensolutions which form a finite dimensional vector space. In this description it is clear that the fluid motions are stable if the real parts of all the eigenvalues have non positive values. It is important to notice that, in this description, since the amplitudes of the eigenmodes are arbitrary, these solutions don't yield information concerning the motion itself.

Our purpose in this paper is to demonstrate the use of the knowledge provided by measurements of the anode currents to determine the amplitudes of the eigenmodes in real situations.

It is in fact rather clear that the sum of the interface fluctuations, corresponding to the different modes (with the amplitudes corresponding to the measurements), describes the time dependent part of the interface motion.

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In a more precise way one can say that, from a physical point of view, the fluctuation of the electric current along one of the anode rods is the sum of the current generated by the different modes. Conversely, the current being known, we can derive the amplitudes of the modes from this information. In this way we can visualize (from these measurements and from the steady solution) the motion of the interface or of any other field. In the case of the interface, since the steady interface is almost flat, the description obtained from the fluctuations alone is already quite good.

The advantage of this method is that it sheds light on the role played by each of the eigenmodes in the description of the motion.

Let us remark that one already knows [6] that the eigenvalues obtained by the solution of the linearized system of equations are in full agreement with the frequencies resulting from a Fourier transform of the anode currents.
From a mathematical point of view we stick to the philosophy followed in [1] to [5]. This means that the eigenmodes and eigenvalues are obtained through a variational formulation of the linearized MHD equations. The effects of turbulence are taken into account by the so-called Moreau-Evans approximation, that is by a damping factor proportional to the velocity field. As will be shown below, the constraint on the fluctuation of the current density which has to match with the current measured along the anode rods appears mathematically as a source term which excites the different modes. More precisely the solution of the system of equations for the electric potential can be expressed as a sum of two terms: a first one corresponding to the effects of the interface and of the velocity field (as in the computation of the eigenmodes and frequencies) and a second one taking into account the constraint mentioned above. These potentials lead to two different current densities and consequently to two distinct force fields; one of them is responsible for the excitation of the eigenmodes.

Since we are dealing with time dependent fields but are using measurements which are obtained from an initial value we perform, on all the equations and conditions, a Laplace transform in the time variable. The final result will evidently be obtained through an inverse Laplace transform which describes the fluctuations of the velocity and of the interface; it is expressed in formula (28). We would like to emphasize the importance of this relation, which allows us to visualize the metal/bath interface as a response to the input constituted by the electric current flowing through the anode rods.

The modeling we are dealing with is briefly described in section two. The equations and conditions resulting from the linearization are also recalled. The problem of the electric potential and of the effects of the constraint on the current are studied in section three. The dynamic problem is described in the next section. Numerical calculations are presented in the last section. The mathematical developments are given in appendices.

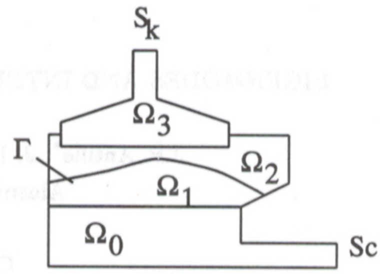

Figure 1 – Half transverse cross section

2. The physical model

As shown in figure 1 and 2, the cell is represented by four domains Ω_j, with boundaries $\partial\Omega_j$, for $j = 0,1,2,3$ (note that $\overline{\Omega}_j = \Omega_j \cup \partial\Omega_j$). We denote by Λ the whole cell, of boundary $\partial\Lambda$ ($\overline{\Lambda} = \Lambda \cup \partial\Lambda$) by Γ the interface (metal surface contour) between the fluids, by Ω the domain occupied by the fluids i.e.

$$\overline{\Lambda} = \cup_{j=0}^{3}\overline{\Omega}_j, \overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \text{ and } \Gamma = \partial\Omega_1 \cap \partial\Omega_2,$$

by S_k $k = 1,2,..,N$ the sections of the anode rods at the height where the current is measured and by S_c the intersection of the boundary of the cell with the cathode.

An arbitrary field $\widetilde{g}$ is written as a sum of two terms i.e.

$$\widetilde{g} = g + G,$$

where g represents the steady field and G the time dependent fluctuation of $\widetilde{g}$ around g. Let G_j be the restriction of a field G to Ω_j. The jump of G through the boundary $\partial\Omega_j \cap \partial\Omega_{j+1}$ is defined by

$$[G]_{\partial\Omega_j \cap \partial\Omega_{j+1}} = G_j - G_{j+1} \text{ for } j = 0 \text{ to } 2.$$

The physical system is described by the interface, current density, velocity, pressure, electric potential and induction fields. They are denoted by $h, \mathbf{j}, \mathbf{u}, p, \phi, \mathbf{b}$ in the steady state. The corresponding fluctuations are expressed by $H, \mathbf{J}, \mathbf{U}, P, \Phi, \mathbf{B}$. We note that the interface is described by the equation

$$x_3 = h(x_1, x_2) + H(x_1, x_2, t). \quad (1)$$

Hydrodynamic equations and conditions.

Setting

$$\mathbf{F}(\mathbf{U},H) = -\rho(\mathbf{u},\nabla)\mathbf{U} - \rho(\mathbf{U},\nabla)\mathbf{u} + \mathbf{J}\wedge\mathbf{b} + \mathbf{j}\wedge\mathbf{B}, \quad (2)$$

and

$$G(\mathbf{U},H) = -(\mathbf{u}, \nabla H) + H(\nabla(x_3 - h), \partial_3\mathbf{u}), \quad (3)$$

the equations for the fluctuations of the velocity and pressure are given in Ω_1 and Ω_2 by

$$\rho\partial_t\,\mathbf{U} = -\nabla P - \rho\kappa\mathbf{U} + \mathbf{F}(\mathbf{U},H), \quad (4)$$

where ρ is the density,

$$div(\mathbf{U}) = 0, \quad (5)$$

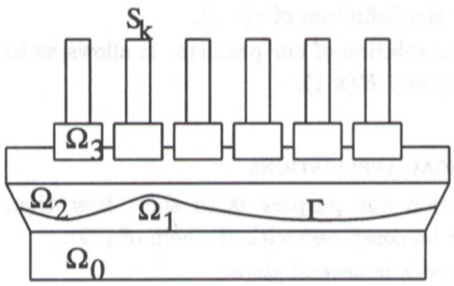

Figure 2 – Longitudinal cross section

whereas on the interface we have

$$\partial_t H = (\nabla(x_3 - h), \mathbf{U}) + G(\mathbf{U}, H) \text{ on } \Gamma \quad (6)$$

with the conditions

$$(\mathbf{U}, \mathbf{n}) = 0 \text{ on } \partial\Omega \text{ and } [P + (\partial_3 p)H]_\Gamma = 0. \quad (7)$$

(see [5] for details)

Electric potential equation and conditions.

For the electric potential we have the equation

$$\sigma\Delta\Phi = \sigma\, div(\mathbf{U} \wedge \mathbf{b} + \mathbf{u} \wedge \mathbf{B}), \quad (8)$$

where σ is the electric conductivity, with the conditions

$$[(\mathbf{J}, \nabla(x_3 - h))]_\Gamma = [(\mathbf{j}, \nabla H) - H(\nabla(x_3 - h), \partial_3 \mathbf{j})]_\Gamma. \quad (9)$$

($(\mathbf{J}, \mathbf{n})$ is continuous on all the other internal boundaries).

On $\partial\Lambda$ the current density satisfies

$$(\mathbf{J}, \mathbf{n}) = \begin{cases} 0 \text{ on } \partial\Lambda \setminus (\cup_{k=1}^{N} S_k \cup S_c) \\ j_0^k \text{ on } S_k, k = 1, 2, .., N, \end{cases} \quad (10)$$

where $\mathbf{n}$ is the outward unit normal to Λ, j_0^k is the current density in the k^{th} rod, the value of which results from the measurements of the current and

$$\mathbf{J} = \begin{cases} -\sigma\nabla\Phi \text{ in } \Omega_0 \text{ and } \Omega_3 \\ \sigma(-\nabla\Phi + \mathbf{U} \wedge \mathbf{b} + \mathbf{u} \wedge \mathbf{B}) \text{ in } \Omega_1 \text{ and } \Omega_2. \end{cases} \quad (11)$$

Furthermore for Φ

$$[\Phi + H\partial_3\phi]_\Gamma = 0 \text{ and } \Phi = 0 \text{ on } S_c. \quad (12)$$

On the other internal boundaries the electric potential is continuous.

The constraints on the current are expressed by the relations

$$(\mathbf{J}, \mathbf{n}) = j_0^k \text{ on } S_k, k = 1, 2, ..., N, \quad (13)$$

where $\mathbf{n}$ is the outward unit normal to Λ.

Magnetic induction equation and conditions.

As shown in [3] the induction field can be expressed by Biot-Savart's law. In other words we have for $\mathbf{B}$ the expression

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int_\Lambda \frac{\mathbf{J}(\mathbf{y}) \wedge (\mathbf{x} - \mathbf{y})}{|(\mathbf{x} - \mathbf{y})|^3} d\tau(\mathbf{y}) + \frac{\mu_0}{4\pi}\int_\Gamma \frac{\alpha H\, [\mathbf{j}]_\Gamma \wedge (\mathbf{x} - \mathbf{y})}{|(\mathbf{x} - \mathbf{y})|^3} d\sigma(\mathbf{y}), \quad (14)$$

where

$$\alpha = \frac{1}{\sqrt{1 + (\partial_1 h)^2 + (\partial_2 h)^2}}. \quad (15)$$

For an arbitrary field f we denote by f_λ its Laplace transform. i.e.

$$f_\lambda(\mathbf{x}) = \int_0^\infty f(\mathbf{x}, t)\exp(-\lambda t)dt,$$

where λ is a complex variable in the domain of convergence.

3. The electric potential.

In this section we introduce the effects of the constraints imposed on the current density in order to match the measured ones along the rods.

We introduce the following notations for the potential and the current density

$$\Phi_\lambda = \Phi'_\lambda + \Phi''_\lambda \text{ and } \mathbf{J}_\lambda = \mathbf{J}'_\lambda + \mathbf{J}''_\lambda. \quad (16)$$

To $\mathbf{J}'_\lambda$ corresponds the induction field $\mathbf{B}'_\lambda$ given by the above Biot-Savart law where $\mathbf{J}'_\lambda$ is substituted to $\mathbf{J}_\lambda$, whereas to $\mathbf{J}''_\lambda$ is associated with the induction field $\mathbf{B}''_\lambda$ expressed by the same law but without the surface integral term. Φ'_λ is the solution of the Laplace transform of the problem for Φ, i.e. equations (8) to (12) but with the assumption that the normal component of the current density is vanishing on $\partial\Lambda$, i.e. $j_0^k = 0$ for all k in (10). In fact Φ'_λ and $\mathbf{B}'_\lambda$ are essentially the same as the fields Φ and $\mathbf{B}$ depending on $\mathbf{U}$ and H, already used in references [1] to [5] in the computation of the eigenmodes and frequencies.

Φ''_λ takes into account the constraints on the current densities. As shown in appendix A it is the solution of a variational problem in the space

$$V'' = \{\psi \in H^1(\Lambda); \psi = 0 \text{ on } S_c\},$$

where $H^1(\Lambda)$ is the classical first order Sobolev space. The problem can be stated as follows.

Find $\Phi''_\lambda \in V''$ satisfying

$$\int_\Lambda \sigma(\nabla\Phi''_\lambda, \nabla\phi)d\tau = \int_\Omega \sigma(\mathbf{u} \wedge \mathbf{B}''_\lambda, \nabla\phi)d\tau + \sum_{k=1}^{N}\int_{S_k}(j_0^k)_\lambda \phi d\sigma \quad \forall\phi \in V''. \quad (17)$$

Let us now consider the Laplace transform of the force field $\mathbf{F}$. From (16) it can be expressed by

$$\mathbf{F}_\lambda = \mathbf{F}'_\lambda + \mathbf{F}''_\lambda, \quad (18)$$

where

$$\mathbf{F}'_\lambda = -\rho(\mathbf{u}, \nabla)\mathbf{U}_\lambda - \rho(\mathbf{U}_\lambda, \nabla)\mathbf{u} + \mathbf{J}'_\lambda \wedge \mathbf{b} + \mathbf{j} \wedge \mathbf{B}'_\lambda \quad (19)$$

and

$$\mathbf{F}''_\lambda = \mathbf{J}''_\lambda \wedge \mathbf{b} + \mathbf{j} \wedge \mathbf{B}''_\lambda, \quad (20)$$

where

$$\mathbf{J}''_\lambda = \sigma(-\nabla\Phi''_\lambda + \mathbf{u} \wedge \mathbf{B}''_\lambda). \quad (21)$$

It is important to notice that $\mathbf{J}''_\lambda$ and $\mathbf{B}''_\lambda$ are depending on the j_0^k only (not on $\mathbf{U}_\lambda$ and H_λ). Conversely one notes that

$\mathbf{J}'_\lambda$ and $\mathbf{B}'_\lambda$ are linear functions in $\mathbf{U}_\lambda$ and H_λ so that for these terms the inverse Laplace transformation consists in performing this operation on $\mathbf{U}_\lambda$ and H_λ only; in other words in replacing $\mathbf{U}_\lambda$ and H_λ by $\mathbf{U}(\mathbf{x},t)$ and $H(\mathbf{x},t)$.

Let $(\Phi_k, \mathbf{B}_k)_{k=0}^N$ be the solutions of the equations (14) (with the first integral only) and of

$$\int_\Lambda \sigma(\nabla\Phi''_\lambda, \nabla\phi)d\tau = \int_\Omega \sigma(\mathbf{u}\wedge\mathbf{B}''_\lambda, \nabla\phi)d\tau + \int_{S_k} \phi d\sigma, \quad (22)$$

for $k = 1, 2, ...N$ (which are special cases of (17)). It is then easy to see, from (17), (20) and (22) that $\mathbf{F}''_\lambda$ can be expressed in the following form

$$\mathbf{F}''_\lambda = \sum_{k=1}^{N} (j_0^k)_\lambda (\mathbf{J}_k \wedge \mathbf{b} + \mathbf{j}\wedge\mathbf{B}_k), \quad (23)$$

an expression which will be used in the next section.

4. The hydrodynamic problem

In order to derive solutions of the system (4) to (7) we first start by performing a Laplace transform on these relations. Taking (18) into account this leads to

$$\rho\lambda\mathbf{U}_\lambda = -\nabla P_\lambda - \rho\kappa\mathbf{U}_\lambda + \mathbf{F}'(\mathbf{U}_\lambda, H_\lambda) + \mathbf{F}''_\lambda + \rho\mathbf{U}(\cdot,0) \quad (24)$$

$$div(\mathbf{U}_\lambda) = 0, \quad (25)$$

$$\lambda H_\lambda = (\nabla(x_3 - h), \mathbf{U}_\lambda) + G(\mathbf{U}_\lambda, H_\lambda) + H(\cdot,0) \text{ on } \Gamma \quad (26)$$

with the conditions

$$(\mathbf{U}_\lambda, \mathbf{n}) = 0 \text{ on } \partial\Omega \text{ and } [P_\lambda + (\partial_3 p)H_\lambda]_\Gamma = 0. \quad (27)$$

This problem has been studied in [4] for the case where $\mathbf{F}''_\lambda + \rho\mathbf{U}_\lambda(\cdot,0)$ and $H_\lambda(\cdot,0)$ are disregarded. In this reference it is shown that (in this case) the solutions, which are obtained with an analytic continuation from the gravitational modes (for non vanishing H) to the MHD ones, the possible frequencies form a discrete set in any bounded set of the complex plane. Moreover the eigensolutions associated with a given element of this set form an finite dimensional vector space. We denote by λ_j the frequency associated to the eigenmode $(\mathbf{U}_j, P_j, H_j)$ for $j = 1, 2, ..., \infty$.

Setting $(z_j)_{j=1}^\infty \equiv (\mathbf{U}_j, H_j)_{j=1}^\infty$ one can show (see appendix B) that for the Galerkin approximation, in the space spanned by the eigenmodes $(z_j)_{j=1}^M$, the general solution has, after a sufficiently large period of time, the following form

$$z(\mathbf{x},t) = \sum_{j=1}^{M} z_j(\mathbf{x})\gamma_j(t)\exp\lambda_j t \quad (28)$$

where

$$\gamma_j(t) = \sum_{k=1}^{N} \int_\Omega (\mathbf{J}_k \wedge \mathbf{b} + \mathbf{j}\wedge\mathbf{B}_k, \overline{\mathbf{w}}_j)d\tau \times \int_0^t j_0^k(t')\exp(-\lambda_j)t'dt', \quad (29)$$

is derived in appendix B and $\widetilde{z}_k \equiv (\mathbf{w}_k, q_k)$ satisfies

$$r(z_j, \widetilde{z}_k) = \delta_{j,k}. \quad (30)$$

(see B4 for the definition of $r(\cdot,\cdot)$).

(28) is the solution of our problem. It allows us to obtain both $\mathbf{U}(\mathbf{x},t)$ and $H(\mathbf{x},t)$.

5. Numerical applications

In this section our purpose is to show how $\mathbf{U}(\mathbf{x},t)$ and $H(\mathbf{x},t)$ can be computed with the help of (28).

One proceeds in several steps.

- One first performs recordings of the anode current fluctuations simultaneously on all the anodes (this gives the set of $j_0^k(t)$, $k = 1, 2, .., N$).

- Making use of the computer program we have developed for the calculation of the eigenmodes and eigenvalues (which is valid for any geometry, not only for the parallelepipedal ones) one obtains the sequence $(z_j, \lambda_j)_{j=1}^M$ of eigenmodes and eigenfrequencies.

- With the help of $r(\cdot,\cdot)$, which is in fact a scalar product on the function space Z (see B_1 to B_4 for the definitions), we compute the set $\widetilde{z}_j = (\mathbf{w}_k, q_k)$ from relation (B10). In fact if $\widetilde{z}_k$ is expressed in the basis formed by the z_j, i.e. if

$$\widetilde{z}_k = \sum_{i=1}^{M} a_k^i z_i \quad (31)$$

(B10) leads, for each fixed j, to the following system of equations

$$\sum_{i=1}^{M} a_k^i r(z_j, z_i) = \delta_{k,j} \text{ for } k = 1, 2, .., M. \quad (32)$$

(32) being solved the $(z_j)_{j=1}^M$ can be obtained.

- From these results (28) is readily computed.

Some computations of the interface for a real cell are given in figure 3.

6. Conclusions

In order to visualize the interface or any other field in the course of time, one can in principle integrate the time dependent MHD equations. However solutions obtained in this way are strongly depending on the initial conditions; moreover it is difficult to get numerical approximations which remain valid for sufficiently large time period.

Since the solutions used in our derivation become rapidely independent of these initial conditions the method presented here, for a visualization of the interface, has the advantage to remain valid for arbitrary time periods. Incidentally it also confirms the accuracy of our computations of the eigenmodes and frequencies.

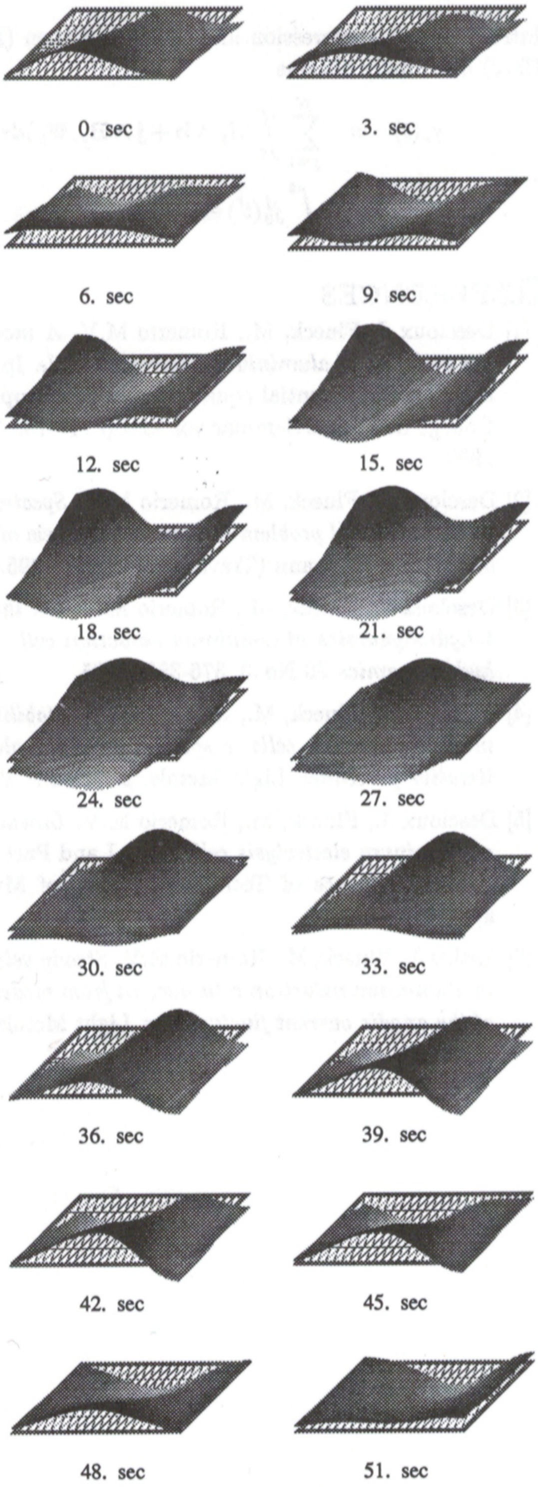

Figure 3 – Displacements of the interface for a sequence of 3 sec. intervals. (4 modes Galerkin approximation)

Appendix A

Our purpose in this appendix is to show that Φ'' satisfies (17).

According to our assumptions Φ' satisfies (8) to (12) but for the fact that $(\mathbf{J},\mathbf{n})=0$ on the S_k, for $k = 1, 2, .., N$ and $\mathbf{B}$ is replaced by $\mathbf{B}'$ in (11).

Since Φ is supposed to fulfill conditions (8) to (12), Φ'' must be a continuous field in the whole cell. It must moreover satisfy

$$\sigma\Delta\Phi'' = \sigma\, div(\mathbf{u}\wedge\mathbf{B}'') \tag{A1}$$

and condition (13).

Integrating (A1) against a sufficiently regular function Ψ one obtains, with Gauss theorem and the conditions just mentioned, the relation

$$\int_\Lambda \sigma(\nabla\Phi'', \nabla\Psi)d\tau = \int_\Lambda \sigma(\mathbf{u}\wedge\mathbf{B}'', \nabla\Psi)d\tau + \sum_{k=1}^{N}\int_{S_k} j_0^k \Psi d\sigma. \tag{A2}$$

Appendix B

Let us introduce some function spaces. We set

$$W = \{\mathbf{w} : \Omega \to (\mathbb{C})^3\,; div(\mathbf{w}) = 0,\ (\mathbf{w},\mathbf{n}) = 0 \text{ on } \partial\Omega\}, \tag{B1}$$

$$\widetilde{W} = \{q : \Gamma \to \mathbb{C}\} \tag{B2}$$

and

$$Z = W \times \widetilde{W}. \tag{B3}$$

In order to introduce a weak formulation, in the space Z, of the problem given by (24) to (27) we defined the following (sesquilinear) forms

$$r(z_\lambda,\xi) = \int_\Omega \rho(\mathbf{U}_\lambda,\overline{\mathbf{w}})d\tau + \int_\Gamma \frac{\alpha^2}{\beta}H_\lambda\overline{q}d\sigma, \tag{B4}$$

$$s(z_\lambda,\xi) = \int_\Gamma \frac{\alpha}{\beta}\{(\mathbf{U}_\lambda,\mathbf{n})\overline{q} - H_\lambda(\overline{\mathbf{w}},\mathbf{n})\}d\sigma$$
$$-\int_\Omega \rho\kappa(\mathbf{U}_\lambda,\overline{\mathbf{w}})d\tau, \tag{B5}$$

$$t(z_\lambda,\xi) = \int_\Omega (\mathbf{F}(z_\lambda),\overline{\mathbf{w}})d\tau + \int_\Gamma \frac{\alpha^2}{\beta}G(z_\lambda)\overline{q}d\sigma \tag{33}$$

where $z_\lambda = (\mathbf{U}_\lambda,H_\lambda)$ and $\xi = (\mathbf{w},q)$ are element of Z and

$$\beta = \frac{-\alpha}{[\partial_3 p]_\Gamma}. \tag{B5}$$

We also introduce the linear form

$$l_\lambda(\xi, z(0)) = \int_\Omega (\mathbf{F}''_\lambda, \overline{\mathbf{w}})d\tau +$$
$$\int_\Omega \rho(\mathbf{U}(\cdot,0),\overline{\mathbf{w}})d\tau + \int_\Gamma \frac{\alpha}{\beta}H(\cdot,0)\overline{q}d\sigma \tag{B7}$$

where $z(\cdot,0) = (\mathbf{U}(\cdot,0),H(\cdot,0))$ is supposed to be given.

Performing the Laplace transform of (4) to (7) and making use of Gauss theorem, one integrates successively the Laplace transforms of (4) against $\overline{\mathbf{w}}$ and of (7) against $\frac{\alpha^2}{\beta}\,\overline{q}$

$((\mathbf{w},q) \in Z)$. Summing these two results one obtains, with (B4) to (B7),

$$\lambda r(z_\lambda,\xi) = s(z_\lambda,\xi) + t(z_\lambda,\xi) + l_\lambda(\xi, z(0)). \quad \text{(B8)}$$

With this relation the problem for the velocity and the interface can be stated in the following weak form

Problem$((U,H)_W)$

For all $\lambda \in \mathbb{C}$ find $z_\lambda \in Z$ satisfying

$$\lambda r(z_\lambda,\xi) = s(z_\lambda,\xi) + t(z_\lambda,\xi) + l_\lambda(\xi, z(0)) \ \ \forall \xi \in Z. \quad \text{(B9)}$$

Let z_j and λ_j, $j = 1, 2, ..., M$ be the eigenmodes and eigenfrequencies, solutions of this equation for $l_\lambda = 0$.

We now look for a Galerkin approximation of Problem$((\mathbf{U}, H)_W)$ in the subspace

$$Z_h = span(z_1, z_2, ... z_M). \quad \text{(B10)}$$

In this purpose we introduce the set of biorthogonal fields $\widetilde{z}_k \equiv (\mathbf{w}_k, q_k) \in Z_h$ defined by the relations

$$r(z_j, \widetilde{z}_k) = \delta_{j,k} \ \ k = 1, 2, ..., M. \quad \text{(B11)}$$

Introducing in (B9) a solution in the form

$$z_\lambda = \sum_{j=1}^{M} \alpha_j z_j \quad \text{(B12)}$$

and taking for ξ the element $\widetilde{z}_k$ defined by (B11) we obtain

$$\lambda \alpha_k = \sum_{j=1}^{M} \alpha_j \left\{ s(z_j; \widetilde{z}_k) + t(z_j; \widetilde{z}_k) \right\} + l_\lambda(\widetilde{z}_k), \ \ k = 1, 2, .., M. \quad \text{(B13)}$$

But since, from the definition of the modes (z_j, λ_j),

$$s(z_j; \widetilde{z}_k) + t(z_j; \widetilde{z}_k) = \lambda_j \delta_{j,k}, \quad \text{(B14)}$$

one can draw from B13 the following expression for z_λ

$$z_\lambda = \sum_{k=1}^{M} \frac{l_\lambda(\widetilde{z}_k)}{\lambda - \lambda_k} z_k. \quad \text{(B15)}$$

One then checks that (B15) is the Laplace transform of

$$z(\mathbf{x}, t) = \sum_{k=1}^{M} z_k(\mathbf{x}) \gamma_k(t) \exp(\lambda_k t) \quad \text{(B16)}$$

where

$$\gamma_k(t) = \int_0^t l(\widetilde{z}_k, t') \exp(-\lambda_k t') dt'. \quad \text{(B17)}$$

In agreement with our assumptions we restrict ourselves to situations where the motion is stable. i.e. for which

$$\mathrm{Re}\, \lambda_k \leq 0 \text{ for } k = 1, 2, .., M. \quad \text{(B18)}$$

This implies that in $l(\widetilde{z}_k, t)$ the terms whith the factors $\mathbf{U}(\cdot, 0)$ or $H(\cdot, 0)$ disappear for some sufficiently large time period; we consequently disregard them. This assumption implies that

$$l(\widetilde{z}_k, t) = \int_\Omega (\mathbf{F}''(\cdot, t), \overline{\mathbf{w}}_k) d\tau. \quad \text{(B19)}$$

The Laplace transform of $\mathbf{F}''_\lambda$ has been obtained in section 3. From (23) one immediately gets

$$\mathbf{F}''(x, t) = \sum_{j=1}^{N} j_0^j(t) (\mathbf{J}_j \wedge \mathbf{b} + \mathbf{j} \wedge \mathbf{B}_j). \quad \text{(B20)}$$

Introducing this expression into (B20) and then (B19) into (B17) one finally obtains

$$\gamma_k(t) = \sum_{j=1}^{N} \int_\Omega (\mathbf{J}_j \wedge \mathbf{b} + \mathbf{j} \wedge \mathbf{B}_j, \overline{\mathbf{w}}_k) d\tau \int_0^t j_0^j(t') \exp(-\lambda_k t') dt'. \quad \text{(B21)}$$

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